01204211 Discrete Mathematics
Lecture 17 (extra): Binomial Coefficients (extra)

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The binomial coefficients[1]

In this lecture, we shall study the function $\binom{n}{k}$ itself.
First, let's see the actual value of the binomial coefficients $\binom{n}{k}$ for various values of n.

[1] This lecture mostly follows Chapter 3 of [LPV].

What do you see?

- The function $\binom{n}{\cdot}$ is symmetric around $n/2$.
- Why? This is true because we know that $\binom{n}{k} = \binom{n}{n-k}$.
- The maximum is at the middle, i.e., when n is even the maximum is at $\binom{n}{n/2}$ and when n is odd, the maximum is at $\binom{n}{\lfloor n/2 \rfloor}$ and $\binom{n}{\lceil n/2 \rceil}$.
- Why? Can we prove that?

Largest in the middle

To understand the behavior of $\binom{n}{k}$ as k changes, let's look at two consecutive values:

$$\binom{n}{k} \heartsuit \binom{n}{k+1}$$

Let's write them out:

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \heartsuit \frac{n(n-1)(n-2)\cdots(n-k)}{(k+1)k!}.$$

Removing common terms, we can see that we are comparing these two terms:

$$1 \heartsuit \frac{n-k}{k+1} \Leftrightarrow k \heartsuit \frac{n-1}{2},$$

that is,

- if $k < (n-1)/2$, $\binom{n}{k} < \binom{n}{k+1}$; and
- if $k > (n-1)/2$, $\binom{n}{k} > \binom{n}{k+1}$.

How large is the middle $\binom{n}{n/2}$

Here, to simplify the calculation, we shall only consider the case when n is even. Let's try to estimate the value of $\binom{n}{n/2}$ by finding its upper and lower bounds.
A simple upper bound can be obtain using the fact that $\binom{n}{n/2}$ counts subsets of certain size:

$$\binom{n}{n/2} < 2^n.$$

We can also get a lower bound by noting that the maximum must be at least the average, i.e.,

$$\binom{n}{n/2} \geq \frac{2^n}{n+1}$$

Combining both bounds, we get that

$$\frac{2^n}{n+1} \leq \binom{n}{n/2} < 2^n.$$

Let's plug in $n = 200$, and calculate the number of digits to see how close these bounds.

$$27.80 \approx 200 \cdot \log 2 - \log 201 \leq \log \binom{n}{n/2} < 200 \cdot \log 2 \approx 30.10$$

Can we get a better approximation?
Yes, with Stirling's formula. (homework)

Concentration

- We know that the maximum of $\binom{n}{k}$ is obtained when $k = n/2$. From the graph, you can see that, as you move further from the middle, the value of the function drops rapidly.
- Since we consider even n, we let $2m = n$. One way to quantify how fast the values drop is to think about the ratio

$$\binom{2m}{m-t} \Big/ \binom{2m}{m}.$$

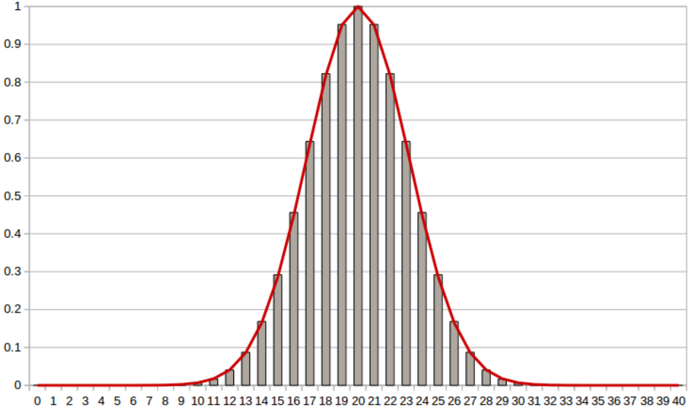
- In fact, it is known that

$$\binom{2m}{m-t} \Big/ \binom{2m}{m} \approx e^{-t^2/m}$$

- We will use our basic tools to obtain weaker bounds.

How close is the approximation?

The estimation $e^{-t^2/m}$ is extremely close as shown in the figure below, where the gray bars are the actual value of $\binom{2m}{m-t}/\binom{2m}{m}$ and the red line is $e^{-t^2/m}$.



The actual values

Because dealing with numbers less than 1 with logarithms is error-prone, we will work on the reciprocal. Let's try to calculate the ratio

$$\binom{2m}{m} \Big/ \binom{2m}{m-t} = \frac{(2m)!}{m!m!} \times \frac{(2m-m+t)!(m-t)!}{(2m)!}$$
$$= \frac{(m+t)(m+t-1)\cdots(m+1)}{m(m-1)(m-2)\cdots(m-t+1)}.$$

We can use the same logarithm trick. We have that the log of the ratio is

$$\ln\left(\frac{m+t}{m}\right) + \ln\left(\frac{m+t-1}{m-1}\right) + \cdots + \ln\left(\frac{m+1}{m-t+1}.\right).$$

Then we can apply the bounds we have for $\ln x$:

$$\frac{x-1}{x} \leq \ln x \leq x-1$$

The upper bound on the reciprocal

Each term in the sum is in this form $\ln((m-i)/(m+t-i))$. Applying the upper bound, we get

$$\ln\left(\frac{m+t-i}{m-i}\right) \leq \frac{m+t-i}{m-i} - 1 = \frac{m+t-i-m+i}{m-i} = \frac{t}{m-i}.$$

Let's sum them up to get

$$\ln\left(\frac{m+t}{m}\right) + \ln\left(\frac{m+t-1}{m-1}\right) + \cdots + \ln\left(\frac{m+1}{m-t+1}.\right)$$

$$\begin{aligned}
&\leq \quad \frac{t}{m} + \frac{t}{m-1} + \cdots + \frac{t}{m-t+1} \\
&\leq \quad \frac{t}{m-t+1} + \frac{t}{m-t+1} + \cdots + \frac{t}{m-t+1} \\
&= \quad \frac{t^2}{m-t+1}.
\end{aligned}$$

This implies that

$$\ln\left(\frac{(m+t)(m+t-1)\cdots(m+1)}{m(m-1)(m-2)\cdots(m-t+1)}\right) \leq \frac{t^2}{m-t+1},$$

i.e.,

$$\begin{aligned}\binom{2m}{m}\Big/\binom{2m}{m-t} &= \left(\frac{(m+t)(m+t-1)\cdots(m+1)}{m(m-1)(m-2)\cdots(m-t+1)}\right)\\ &\leq e^{t^2/(m-t+1)}.\end{aligned}$$

Taking the reciprocal, we get

$$e^{-t^2/(m-t+1)} \leq \binom{2m}{m-t}\Big/\binom{2m}{m}.$$

Upper bounds

Using the same approach, we can show that

$$\binom{2m}{m-t} \Big/ \binom{2m}{m} \leq e^{-t^2/(m+t)}.$$

Thus, we derived the estimates:

$$\boxed{e^{-t^2/(m-t+1)} \leq \binom{2m}{m-t} \Big/ \binom{2m}{m} \leq e^{-t^2/(m+t)},}$$

which is fairly close the the estimate of $e^{-t^2/m}$.

How fast?

- Let's return to the question on how fast do the values of the binomial coefficients decrease as you move further from the middle. Let's use the better estimate $\binom{2m}{m-t}/\binom{2m}{m} \approx e^{-t^2/m}$.
- Given a constant C, we want to estimate the value of t such that $\binom{2m}{m-t}$ is less than $\binom{2m}{m}/C$. (E.g., we can set $C = 2$ to see when the value drops by 50%.) Therefore, we want to find t such that

$$1/C \geq \binom{2m}{m-t} \Big/ \binom{2m}{m} \approx e^{-t^2/m}$$

Taking the logs, we get

$$\ln 1/C = -\ln C \geq \ln \binom{2m}{m-t} \Big/ \binom{2m}{m} \approx -t^2/m.$$

This is true when

$$t \geq \sqrt{m \ln C}.$$

What does this means?

As an example, let $m = 20$ and $C = 2$. We know that when t is approximately $\sqrt{20 \cdot \ln 2} = 3.723$ the value of $\binom{2m}{m-t}$ drops by 50%.