01204211 Discrete Mathematics
Lecture 7d: Fibonacci sequence

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The Fibonacci sequence[1]

Source:

https://en.wikipedia.org/wiki/

File:Fibonacci.jpg

In 1202, Leonardo Bonacci (known as Fibonacci) asked the following question.

> "[A]ssuming that: a newly born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on."
>
> "The puzzle that Fibonacci posed was: how many pairs will there be in one year?"

From https://en.wikipedia.org/wiki/Fibonacci_number

[1] This lecture mostly follows Chapter 4 of [LPV].

Let's try to solve Fibonacci's question.

Let ♠ denote a newly born rabit pair, and ♡ denote a mature rabit pair.

Month	Rabits	
1	♠	1
2	♡	1
3	♡ ♠	2
4	♡ ♡ ♠	3
5	♡ ♡ ♡ ♠ ♠	5
6	♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠	8
7	♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ ♠ ♠ ♠ ♠ ♠	13

How many rabit pairs do we have at the beginning of the 8th month?

- Surely all 13 rabit pairs we have in the 7th month remain there and are all mature. So, the question is how many newly born rabbit pairs that we have.
- The number of newly born rabbit pairs equals the number of mature rabbit pairs we have. This is also equal to the number of rabit pairs that we have in the 6th month: 8.

Thus, we will have 13+8 rabit pairs at the beginning of the 8th month.
If we write down the sequence, we get the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Again, what's the next number in this sequence? How can you compute it?
21+13 = 34 is the answer. You take the last two numbers and add them up to get the next number. Why?

To be precise, let F_n be the n-th number in the Fibonacci sequence. (That is, $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$ and so on.) We can define the $(n+1)$-th number as

$$F_{n+1} = F_n + F_{n-1},$$

for $n = 2, 3, \ldots$. Is this enough to completely specify the sequence? No, because we do not know how to start. To get the Fibonacci sequence, we need to specify two starting values: $F_1 = 1$ and $F_2 = 1$ as well.

Now, you can see that the equation and these special values uniquely determine the sequence. It is also convenient to define $F_0 = 0$ so that the equation works for $n = 1$.

A recurrence

The equation

$$F_{n+1} = F_n + F_{n-1}$$

and the initial values $F_0 = 0$ and $F_1 = 1$ specify all values of the Fibonacci sequence. With these two initial values, you can use the equation to find the value of any number in the sequence.

This definition is called a **recurrence**. Instead of defining the value of each number in the sequence explicitly, we do so by using the values of other numbers in the sequence.

Tilings with 1x1 and 2x1 tiles

You have a walk way of length n units. The width of the walk way is 1 unit. You have unlimited supplies of 1x1 tiles and 2x1 tiles. Every tile of the same size is indistinguishable. In how many ways can you tile the walk way?
Let's consider small cases.

- When $n = 1$, there are 1 way.
- When $n = 2$, there are 2 ways.
- When $n = 3$, there are 3 ways.
- When $n = 4$, there are 5 ways.

Let's define J_n to be the number of ways you can tile a walk way of length n. From the example above, we know that $J_1 = 1$ and $J_2 = 2$.
Can you find a formula for general J_n?

Figuring out the recurrence for J_n

To figure out the general formula for J_n, we can think about the first choice we can make when tiling a walk way of length n. There are two choices:

- (1) We can start placing a 1x1 tile at the beginning, or
- (2) We can start placing a 2x1 tile at the beginning.

In each of the cases, let's think about how many ways we can tile the rest of the walk way, provided that the first step is made.

Note that if we start by placing a 1x1 tile, we are left with a walk way of length $n-1$. From the definition of J_n, we know that there are J_{n-1} ways to tile the rest of the walk way of length $n-1$. Using similar reasoning, we know that if we start with a 2x1 tile, there are J_{n-2} ways to tile the rest of the walk way.

The recurrence for J_n

From the discussion, we have that

$$J_n = J_{n-1} + J_{n-2},$$

where $J_1 = 1$ and $J_2 = 2$.

Note that this is exactly the same recurrence as the Fibonacci sequence, but with different initial values. In fact, we have that

$$J_n = F_{n+1}.$$

Identities on Fibonacci numbers

There are a lot of identities related to Fibonacci numbers. Let's see the first few values in the sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Now, let's add the first few numbers:

$$\begin{aligned}
0+1 &= 1 \\
0+1+1 &= 2 \\
0+1+1+2 &= 4 \\
0+1+1+2+3 &= 7 \\
0+1+1+2+3+5 &= 12 \\
0+1+1+2+3+5+8 &= 20 \\
0+1+1+2+3+5+8+13 &= 33
\end{aligned}$$

From this we can formulate the following conjecture:

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1.$$

Theorem: For $n \geq 0$, we have that

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1.$$

Proof: We shall prove by induction on n. The base case has already been demonstrated when we consider small values of n.

Inductive Step: Let's assume that the statement is true for $n = k$, for $k \geq 0$, i.e., assume that

$$F_0 + F_1 + \cdots + F_k = F_{k+2} - 1.$$

We shall prove that the statement is true when $n = k + 1$. This is not hard to show. We write

$$\begin{aligned} (F_0 + F_1 + \cdots + F_k) + F_{k+1} &= (F_{k+2} - 1) + F_{k+1} \\ &= F_{k+3} - 1, \end{aligned}$$

as required. Note that the first step follows from the induction hypothesis. ■