

01204211 Discrete Mathematics  
Lecture 7b: Binomial Coefficients (2)

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# The binomial coefficients<sup>1</sup>

There is a reason why the term  $\binom{n}{k}$  is called the binomial coefficients. In this lecture, we will discuss

- ▶ identities on binomial coefficients.

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<sup>1</sup>This lecture mostly follows Chapter 3 of [LPV].

## Identities in the Triangle

A Pascal's Triangle with 6 rows, enclosed in a rounded rectangle. The numbers are arranged in a triangular pattern, with each number being the sum of the two numbers directly above it. The values are: Row 0: 1; Row 1: 1, 1; Row 2: 1, 2, 1; Row 3: 1, 3, 3, 1; Row 4: 1, 4, 6, 4, 1; Row 5: 1, 5, 10, 10, 5, 1.

						1										
					1		1									
			1		2		1									
		1		3		3		1								
	1		4		6		4		1							
1		1		5		10		10		5		1				
	1		6		15		20		15		6		1			
		1		7		21		35		35		21		7		1

## Odd and even subsets

						1									
						1		1							
				1		2		1							
			1		3		3		1						
		1		4		6		4		1					
	1		5		10		10		5		1				
1		1		6		15		20		15		6		1	
	1		7		21		35		35		21		7		1

Let's try to prove this identity with the Pascal's triangle

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \dots + (-1)^n \binom{n}{n} = 0.$$

## A more formal proof

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0.$$

## The next experiment

						1								
					1		1							
				1		2		1						
			1		3		3		1					
		1		4		6		4		1				
	1		1	5		10		10		5		1		
		1	6		15		20		15		6		1	
1			7		21		35		35		21		7	1

Let's try to compute the sum of squares of numbers in each row.

$$1^2 = 1$$

$$1^2 + 1^2 = 2$$

$$1^2 + 2^2 + 1^2 = 6$$

$$1^2 + 3^2 + 3^2 + 1^2 = 20$$

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$$

Theorem:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

## Another identity

							1							
								1						
							1	2	1					
						1	3	3	1					
					1	4	6	4	1					
			1		5	10	10	5	1					
		1	6		15	20	15	6	1					
	1	7	21		35	35	21	7	1					
1	7	21	35		35	21	7	1						

This suggests

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$



Theorem:

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$