

01204211 Discrete Mathematics
Lecture 7a: Binomial Coefficients (1)

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The binomial coefficients¹

There is a reason why the term $\binom{n}{k}$ is called the binomial coefficients. In this lecture, we will discuss

- ▶ the Pascal's triangle,
- ▶ the binomial theorem

¹This lecture mostly follows Chapter 3 of [LPV].

The equation

Last time we proved that, for $n, k > 0$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

While we can prove this equation algebraically using definitions of binomial coefficients, proving the fact by describing the process of choosing k -subsets reveals interesting insights. This equation also hints us how to compute the value of $\binom{n}{k}$ using values of $\binom{n-1}{\cdot}$'s. So, let's try to do it.

The table

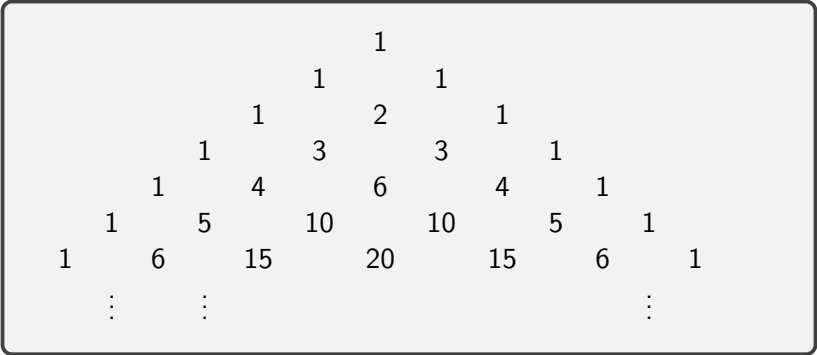
We shall use the fact that $\binom{n}{0} = 1$ and $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ to fill in the following table.

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|----|----|----|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

You can note that the table is left-right symmetric. This is true because of the fact that $\binom{n}{k} = \binom{n}{n-k}$.

The Triangle

If we move the numbers in the table slightly to the right, the table becomes the Pascal's triangle.



A Pascal's triangle with 10 rows of numbers, where each row is shifted one position to the right relative to the row above it. The numbers are arranged in a triangular shape. The first row has one '1'. The second row has two '1's. The third row has three numbers: 1, 2, 1. The fourth row has four numbers: 1, 3, 3, 1. The fifth row has five numbers: 1, 4, 6, 4, 1. The sixth row has six numbers: 1, 5, 10, 10, 5, 1. The seventh row has seven numbers: 1, 6, 15, 20, 15, 6, 1. The eighth row has eight numbers: 1, 7, 21, 35, 35, 21, 7, 1. The ninth row has nine numbers: 1, 8, 28, 56, 70, 56, 28, 8, 1. The tenth row has ten numbers: 1, 9, 36, 84, 126, 126, 84, 36, 9, 1. Vertical ellipses are placed below the first, second, and eighth numbers of the tenth row.

| | | | | | | | | | | |
|--|---|---|----|----|-----|-----|----|----|---|---|
| | | | | | | | | | | 1 |
| | | | | | | | | | 1 | 1 |
| | | | | | | | | 1 | 2 | 1 |
| | | | | | | | 1 | 3 | 3 | 1 |
| | | | | | | 1 | 4 | 6 | 4 | 1 |
| | | | | | 1 | 5 | 10 | 10 | 5 | 1 |
| | | | | 1 | 6 | 15 | 20 | 15 | 6 | 1 |
| | | | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |
| | | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 |
| | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 |
| | ⋮ | ⋮ | | | | | | | ⋮ | |

The table and the binomial coefficients have many other interesting properties.

Polynomial expansions

Let's start by looking at polynomial of the form $(x + y)^n$. Let's start with small values of n :

▶ $(x + y)^1 = x + y$

▶ $(x + y)^2 = x^2 + 2 \cdot xy + y^2$

▶ $(x + y)^3 = x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + y^3$

▶ $(x + y)^4 = x^4 + 4 \cdot x^3y + 6 \cdot x^2y^2 + 4 \cdot xy^3 + y^4.$

Let's focus on the coefficient of each term. You may notice that terms x^n and y^n always have 1 as their coefficients. *Why is that?* Let's look further at the coefficients of terms $x^{n-1}y$. Do you see any pattern in their coefficients? *Can you explain why?*

Another way to look at it

Let's take a look at $(x + y)^4$ again. It is

$$(x + y)(x + y)(x + y)(x + y).$$

- ▶ How do we get x^4 in the expansion? For every factor, you have to pick x .
- ▶ How do we get x^3y in the expansion? Out of the 4 factors, you have to pick y in one of the factor (or you have to pick x in 3 of the factors). Thus there are $\binom{4}{3} = \binom{4}{1}$ ways to do so.

The binomial theorem

Theorem: If you expand $(x + y)^n$, the coefficient of the term $x^k y^{n-k}$ is $\binom{n}{k}$.

That is,

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} =$$
$$\binom{n}{n} x^n + \binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n-2} x^{n-2} y^2 + \dots + \binom{n}{1} x y^{n-1} + \binom{n}{0} y^n.$$

Additional applications of the binomial theorem

The binomial theorem can be used to prove various identities regarding the binomial coefficients. For example, if we let $x = 1$ and $y = 1$, we get that

$$(1 + 1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Quick check. Can you prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0.$$

Note that this statements says that the number of odd subsets equals the number of even subsets.