01204211 Discrete Mathematics
Lecture 6b: Counting 4

Jittat Fakcharoenphol

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Quick practice

Theorem: For a non-empty set, the number of subsets whose sizes are odd equals the number of subsets whose sizes are even. I.e., for $n > 0$,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots .$$

Quick questions (1)

There are 40 students in the classroom. There are 35 students who like Naruto, 10 students who like Bleach, and 7 students who like both of them. How many students in this classroom who do not like either Bleach or Naruto?

Quick questions (2)

There are 35 students in the classroom. There are 25 students who like Naruto, 15 students who like Bleach, 12 students who like One Piece. There are 10 students who like both Naruto and Bleach, 7 students who like both Bleach and One Piece, and 9 students who like both Naruto and One Piece. There are 5 students who like all of them.
How many students in this classroom who do not like any of Bleach, Naruto, or One Piece?

Is this correct?

The answer from the previous quick question is

$$35 - (25 + 15 + 12 - 10 - 7 - 9 + 5) = 4.$$

Is this correct? Why?

Let's try to argue that this answer is, in fact, correct and try to find general answers to this kind of counting questions.

Let's look at an individual student (1)

			N	B	O	NB	BO	NO	NBO	
		35	-25	-15	-12	$+10$	$+7$	$+9$	-5	4
Alfred	N,O	*	*		*			*		
Bobby	B	*		*						
Cathy	B,O	*		*	*		*			
Dave	N,B,O	*	*	*	*	*	*	*	*	
Eddy	-	*								
⋮	⋮									

Let's look at an individual student (2)

			N	B	O	NB	BO	NO	NBO	
		35	-25	-15	-12	$+10$	$+7$	$+9$	-5	4
Alfred	N,O	1	-1		-1			$+1$		0
Bobby	B	1		-1						0
Cathy	B,O	1		-1	-1		$+1$			0
Dave	N,B,O	1	-1	-1	-1	$+1$	$+1$	$+1$	-1	0
Eddy	-	1								1
$\vdots$	$\vdots$									

Let's see how each one is counted

Alfred (N,O):

$$1 - \binom{2}{1} + \binom{2}{2} = 1 - 2 + 1 = 0$$

Bobby (B):

$$1 - \binom{1}{1} = 1 - 1 = 0$$

Dave (N,B,O):

$$1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 1 - 3 + 3 - 1 = 0$$

Do you see any patterns here? How about

$$1 - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5} \quad ?$$

Underlying structures

Let's write 1 as $\binom{5}{0}$. Also, let's separate plus terms and minus terms:

$$\binom{5}{0} + \binom{5}{2} + \binom{5}{4} \quad \heartsuit \quad \binom{5}{1} + \binom{5}{3} + \binom{5}{5}$$

Note that the left terms are the number of even subsets and the right terms are the number of odd subsets. Do you recall what we have done at the beginning of this lecture? We have proved this:

Theorem: The number of even subsets is equal to the number of odd subsets.

This theorem also shows that our calculation technique is correct. This technique is usually called the **Inclusion-Exclusion principle**.