01204211 Discrete Mathematics
Lecture 5b: Counting 1

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Let's count

Club representatives: You are a second year student. Your board game club has 40 members which are in the first year. There is a big competition very soon, so the club has to find exactly 2 representatives (from the first-year students) for the competition.

- How to find these 2 representatives? One of your friends suggests that to be fair to everyone, you have to look at every possible pair and see how the 2 members of the pair play together as a team.
- It might take a very long time, you think. How many pairs are there?

Club representatives (1)

- To choose the member of the pair, you pick the first member and then pick the second member.
- There are 40 ways to choose the first member. For every person you pick as the first member, there are exactly 39 left to pick as the second one. Therefore, there are $40 \cdot 39 = 1,560$ ways.
- Wait.. This is over counting. Picking a as the first member and b as the second member results in the same pair as picking b first and a second. Thus, every pair is counted twice.
- The correct number of pairs is 780; too many possibilities to consider, you conclude.

Club representatives (2)

- Since 780 is too many, you decide to randomly choose 15 pairs of representatives and observe how each pair plays.
- Your friend argues that 15 is too small. Because the number of members is 40 and we will miss someone there.
- So you ask, how many pairs one have to randomly choose from 40 members so that it is very likely that every member is picked once?
- You try to calculate the number, but your friend starts writing a program to simulate.

Club representatives (2)

- Here's the table of the simulation. For each value of number of random pairs, 2,000 simulations has been conducted.

Number of pairs to random	% of choosing everyone once
20	0.00
30	0.00
40	0.15
50	2.45
60	12.05
80	51.65
100	78.00
120	91.25
140	97.10

- You end up choosing randomly 100 pairs, as it has about 80% chance. You feel so tired, but you keep wondering if you can calculate the number without having to write a program.

Club representative again (1)

A team: Another team competition is coming up. It requires a team of 5 players. In the team, each player can play either as Protoss, Terrans, or Zerg. Luckily, only one team of 5 members volunteers to participate.

- To find the best team organization, you ask them to try all possible configurations of race choices against AI players. How many games do you have to watch?
- Each member has 3 choices and this member's choice is independent of the other. Therefore, there are $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$ possible ways.
- You are still tired from watching 100 pairs of players. So you change your mind and ask them to try only configurations that contain all the three races. How many are there?

Club representative again (2)

- Your old friend asks you if you need a computer program. You say 'No', and try to think of the answer.
- You come up with an idea. Instead of counting "good configurations", let's count "bad" ones, i.e., configurations that miss some race.
  - Configurations containing one race: 3
  - Configurations containing only two races, say, Protoss and Terrans: each person has 2 choices; therefore, there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$ configurations.
  - There are 3 ways to have two races. Therefore, there are $32 \cdot 3 = 96$ ways.
  - Thus, the number of good configurations is $243 - 96 - 3 = 144$ ways.
- That's not too many. So you let them play 144 games. It turns out that 144 is wrong.
- Can you spot the mistake?

Club representative again (3)

- Again, we over-count some configuration, e.g., the configuration where all players play Protoss is counted 3 times.
- In fact, it appears the first time we count single race configurations, appears when we count Protoss-Terrans, and finally appears when we count Protoss-Zerg.
- The number of bad configurations is $96 - 3$.
- The correct number of games would be $243 - (96 - 3) = 150$.

Sets: quick review (1)

- Sets are very important notions in mathematics. A **set** is a collection of elements.
- Common set of numbers: real numbers $\mathbb{R}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, positive integers $\mathbb{N}$.
- There are many ways to specify sets.
  - By listing all elements: $\{2, 3, 5, 7, 11\}$
  - By describing its elements: {all prime numbers}
  - By filtering elements from other sets: $\{p \in \mathbb{N} : p$ is a prime$\}$.

Sets: quick review (2)

- If a is an element of S, we write $a \in S$. The **cardinality** of a set is the number of its elements. We denote by $|A|$, the cardinality of A.
- Note that $|\{2, 3, 5, 7, 11\}| = 5$ and $|\mathbb{Z}| = \infty$. A set whose cardinality is zero is called an **empty set**, denoted by $\emptyset$.
- If every element of A is also an element of B, we say that A is a **subset** of B, denoted by $A \subseteq B$. For example,
  - $\{1, 3\} \subseteq \{1, 2, 3, 4\}$
  - $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
  - $\{p \in \mathbb{N} : p \text{ is prime and } p > 2\} \subseteq \{x \in \mathbb{N} : x \text{ is odd}\}$

  Note that $A \subseteq A$ and $\emptyset \subseteq A$.
- If $A \subseteq B$ but $A \neq B$, we write $A \subset B$.

Set operations

Suppose that we are given two sets A and B.

- An **intersection**, denoted by $A \cap B$, is a set whose elements are elements of both A and B.
- A **union**, denoted by $A \cup B$, is a set whose elements are elements of A or B.
- A **difference** of A and B, denoted by $A \setminus B$ or $A - B$, is a set whose elements are elements of A but not elements of B.

Note that

- $A \cap B \subseteq A$
- $A \subseteq A \cup B$
- $A \setminus B \subseteq A$
- If $A \subseteq B$, then $A \setminus B = \emptyset$.

The number of subsets[1]

- Let's get back to counting. Our next question: *What is the number of all subsets of a set with n elements?*
- There are many ways to figure out the answer. Sometimes it is useful to start by looking at small examples.
- Note that the actual set itself does not matter, i.e., the numbers of subsets of $\{1, 2, 3, 4\}$ and $\{a, b, c, d\}$ are equal.
  - $\emptyset$ has 1 subset: itself.
  - $\{1\}$ has 2 subsets: $\emptyset$ and $\{1\}$.
  - $\{1, 2\}$ has 4 subsets.
  - $\{1, 2, 3\}$ has 8 subsets.
- We can guess that the answer is 2^n. But how can we prove that?

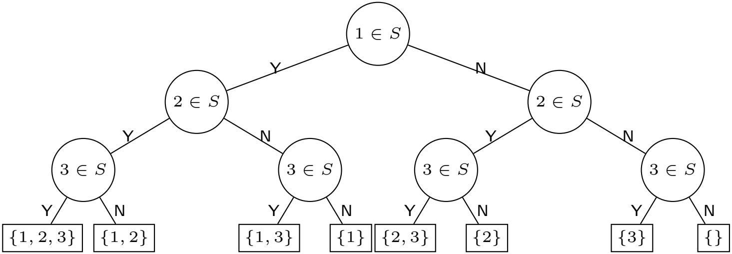
[1]Materials on counting mostly follows [LPV].

Choosing a subset (1)

- Let's try to think about how to select a subset. To be concrete, let's consider choosing a subset of set $\{1, 2, 3\}$.
- There are many ways to do so, but let's follow an obvious one: consider each element and make a decision.
  - 1st step: we have a choice of choosing 1 or not choosing 1: 2 choices.
  - 2nd step: let's consider 2. Whatever the decision that we make on 1, we have 2 choices for 2.
  - 3rd step: let's consider 3. Whatever the decision that we make on 1 and 2, we again have 2 choices for 3.
- This concludes that we have, in total, $2 \cdot 2 \cdot 2 = 8$ ways of choosing subsets of set $\{1, 2, 3\}$.

A decision tree

We describe the process as a decision tree for choosing subset S from $A = \{1, 2, 3\}$.



Choosing a subset (2)

- To be concrete, let's consider choosing a subset of set $\{1, 2, 3\}$.
- We make 3 decisions, for all elements in the set. The number of ways we can choose a subset is 8.
- While we know that it is the correct answer, let's look back on what we are trying to do.
- We are trying to count the number of subsets. But we are actually counting the number of ways we can choose a subset.
- To make sure that they are the same number, we need to make sure that:
  - We count everything: for every possible subset, there is at least one way we can choose it.
  - We do not over count: any two different ways of choosing subsets produce two different subsets.

The number of subsets

Let's think about a general case for set $A = \{1, 2, \ldots, n\}$ with n elements. We choose elements of a subset in n steps. Each step i, for $1 \leq i \leq n$, we consider $i \in A$, and we have 2 choices. Hence, there are 2^n ways of choosing subsets.

- **To see that we can choose every subset:** consider a given subset $S \subseteq A$. Note that we when we consider i, we can choose i if and only if $i \in S$.
- **To see that two different ways of choosing subsets produce different subsets:** consider two different ways of choosing subsets. Since they differ in some step, let $i \in A$ be the element that they choose differently. Thus, one of the subsets contain i while the other does not.

Therefore, we have proved the following theorem.

Theorem: The number of subsets of a set with n elements is 2^n.