01204211 Discrete Mathematics
Lecture 4c: Mathematical Induction 3

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Review: Mathematical Induction

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts:
Base case: $P(1)$
Inductive step: For any $k \geq 1$, $P(k) \Rightarrow P(k+1)$

The **Principle of Mathematical Induction** states that $P(n)$ is true for every natural number n.

The assumption $P(k)$ in the inductive step is usually referred to as **the Induction Hypothesis**.

The Induction Hypothesis

Theorem 1

For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$.

Proof.

The statement $P(n)$ that we want to prove is
"$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$".

Case case: For $n = 1$, the statement is true because $1 < 2$.

Inductive step: For $k \geq 1$, let's assume $P(k)$ and we prove that $P(k+1)$ is true.
The induction hypothesis is: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2$.
We want to show $P(k+1)$, i.e.,
$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2$.
Then... $\square$

Strengtening the Induction Hypothesis (1)

- Is the assumption

  $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2.$$

  "strong" enough to prove

  $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 \quad ?$$

  Why?
- To prove $P(k+1)$, we need a "gap" between the LHS and 2, so that we can add $1/(k+1)$ without blowing up the RHS.

Strengtening the Induction Hypothesis (2)

- Let's see a few values of the sum:
  - $1/1 = 1$.
  - $1/1 + 1/4 = 1.25$.
  - $1/1 + 1/4 + 1/9 \approx 1.361$.
  - $1/1 + 1/4 + 1/9 + 1/16 \approx 1.4236$.
  - $1/1 + 1/4 + 1/9 + 1/16 + 1/25 \approx 1.4636$.

  Yes, there is a gap. But how large?
- We need the gap to be large enough to insert $1/(k+1)^2$.
- After a "mysterious" moment, we observe that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Strengtening the Induction Hypothesis (3)

Theorem 2

For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

Proof.

(... the beginning is left out ...)
Inductive step: For $k \geq 1$, assume that $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$.
Adding $1/(k+1)^2$ on both sides, we get

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right).$$

Since $1/k - 1/(k+1) = 1/(k(k+1))$, we have that

$$1/(k+1) = 1/k - 1/(k(k+1)) < 1/k - 1/(k+1)^2.$$

Therefore, we conclude that

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) \leq 2 - \frac{1}{k+1},$$

as required. $\square$

A Lesson learned

- Is a stronger statement easier to prove?
- In this case, the statement is indeed stronger, but the induction hypothesis gets stronger as well. Sometimes, this works out nicely.