01204211 Discrete Mathematics
Lecture 4b: Mathematical Induction 2

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Review: Mathematical Induction

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts:
Base case: $P(1)$
Inductive step: For any $k \geq 1$, $P(k) \Rightarrow P(k+1)$

The **Principle of Mathematical Induction** states that $P(n)$ is true for every natural number n.

The assumption $P(k)$ in the inductive step is usually referred to as **the Induction Hypothesis**.

Example 1

Theorem: For every natural number n,
$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$

Proof: We prove by induction. The property that we want to prove $P(n)$ is "$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$."

Base case: We can plug in $n = 1$ to check that $P(1)$ is true:
$1^2 = \frac{1}{6}(1+1)(2 \cdot 1 + 1)$.

Inductive step: We assume that $P(k)$ is true for $k \geq 1$ and show that $P(k+1)$ is true.

We first assume the Induction Hypothesis $P(k)$:
$\sum_{i=1}^{k} i^2 = \frac{k}{6}(k+1)(2k+1)$

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Example 1 (cont.)

Let's show $P(k+1)$. We write $\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2$.

Using the Induction Hypothesis, we know that this is equal to

$$\begin{aligned}
(k/6)(k+1)(2k+1) + (k+1)^2 &= \frac{(k+1)}{6}(k(2k+1) + 6(k+1)) \\
&\quad \text{(In this step, we factor out } (k+1)/6\text{)} \\
&= \frac{(k+1)}{6}(2k^2 + 7k + 6) \\
&= \frac{(k+1)}{6}((k+1)+1)(2(k+1)+1).
\end{aligned}$$

This implies $P(k+1)$ as required.
From the Principle of Mathematical Induction, this implies that $P(n)$ is true for every natural number n. ■

Not an example (1)

Theorem 1

For any set of cows, all cows have the same color.

Proof.

We prove by induction on the size n of the set of cows.

Base case: For $n = 1$, clearly for any set of a single cow, every cow in the set has the same color.

Inductive step: Suppose that for every set of size k of cows, all cows in the set have the same color.

We will show that every set of size $k + 1$ of cows, all cows in this set have the same color.

Not an example (2)

Inductive step (cont.): Consider set A of $k+1$ cows.

Because we have established that the base case and the inductive step is true, we can conclude that for any set of cows, all cows have the same color. $\blacksquare$

Not an example (3)

Clearly the following theorem cannot be true.

Theorem 2

For any set of cows, all cows have the same color.

What is wrong with its proof based on mathematical induction?

Unused facts

- Let's informally think about how proving $P(1)$ and $P(k) \Rightarrow P(k+1)$ for all $k \geq 1$ implies that $P(n)$ is true for all natural number n.

- One may notice that when we prove a statement $P(n)$ for all natural number n by induction, during the inductive step where we want to show $P(k+1)$ from $P(k)$, we usually have that $P(1), P(2), \ldots, P(k)$ is true at hands as well.
- Then why don't we use them as well?

Strong Mathematical Induction

Strong Induction

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts: **Base case:** $P(1)$
Inductive step: For any $k \geq 1$,

$$P(1) \wedge P(2) \wedge \cdots \wedge P(k) \Rightarrow P(k+1).$$

Then $P(n)$ is true for every natural number n.

Example 2

Theorem: For any integer $n \geq 4$, one can use only 2-baht coins and 3-baht coins to obtain exactly n baht.

Proof: We prove by strong induction on n.
Base cases: For $n = 4$, we can use two 2-baht coins. For $n = 5$, we can use one 2-baht coin and one 3-baht coin.

Inductive step: Assume that for $k \geq 5$, we can obtain exactly ℓ baht, for $4 \leq \ell \leq k$, using only 2-baht and 3-baht coins. We will show how to obtain a set of $k + 1$ baht.
Since $k \geq 5$, we have that $k - 1 \geq 4$. Therefore from the Induction Hypothesis, we can use only 2-baht coins and 3-baht coins to form a set of coins of total value $k - 1$ baht. With one additional 2-baht coin, we can obtain a set of value $(k - 1) + 2 = k + 1$ baht, as required.

From the Principle of Strong Mathematical Induction, we conclude that the theorem is true. ■

Is strong induction more powerful?

- Can we prove the previous theorem without using the strong induction? Yes, you can (homework).
- In fact, if you can prove that $P(n)$ is true for all natural number n with strong induction, you can always prove it with mathematical induction.
- Hint: Let $Q(n) = P(1) \wedge P(2) \wedge \cdots \wedge P(n)$.