01204211 Discrete Mathematics
Lecture 4a: Mathematical Induction 1

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Mathematical Induction

- In this lecture, we will focus on how to prove properties on natural numbers.
- For example, we may want to prove that for any integer $n \geq 1$,

$$\sum_{i=1}^{n} i = n(n+1)/2,$$

or for any integer $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1),$$

or "We can pay any integer amount $x \geq 4$ baht with 2-baht coins and 5-baht coins."

A review of the summation notation (by examples)

- $\sum_{i=1}^{10} i = 1 + 2 + \cdots + 10.$
  (reads "sum from $i = 1$ to 10 of i" or "sum of i from $i = 1$ to 10")
- $\sum_{i=7}^{9} (i^2 + i) = (7^2 + 7) + (8^2 + 8) + (9^2 + 9).$
  (reads "sum from $i = 7$ to 9 of $i^2 + i$" or "sum of $i^2 + i$ from $i = 7$ to 9")
- The range of the index may be sets. For example, let $A = \{1, 2, 4, 15\}$, we have that $\sum_{i \in A} i^2 = 1^2 + 2^2 + 4^2 + 15^2$.
- What is $\sum_{i=5}^{2} i$? Note that in this case, the range is empty. This sum is called an **empty sum**. By convention, we define it to be zero.

Informal arguments (1)

- Let's try to check that $\sum_{i=1}^{n} i = n(n+1)/2$, for any integer $n \geq 1$, by experimentation.
- Try $n = 1$: LHS[1]: 1, RHS[2]: $1(1+1)/2 = 1$, OK
- Try $n = 2$: LHS: $1 + 2 = 3$, RHS: $2(2+1)/2 = 3$, OK
- Try $n = 3$: LHS: $1 + 2 + 3 = 6$, RHS: $3(3+1)/2 = 6$, OK
- Try ...
- With this trying-all approach, we can't actually prove this statement.

[1] LHS = left hand side
[2] RHS = right hand side

Informal arguments (2)

- Our goal is to show that $\sum_{i=1}^{n} i = n(n+1)/2$, for any integer $n \geq 1$.
- Try $n = 2$: LHS: $1 + 2 = 3$, RHS: $2(2+1)/2 = 3$.
- Try $n = 3$: LHS: $1 + 2 + 3$, RHS: $3(3+1)/2$
- If we compare these two lines, we can see that

$$\begin{aligned} 1 + 2 + 3 &= (1 + 2) + 3 \\ &= 2(2+1)/2 + 3 \qquad \text{(*)} \\ &= 2(2+1)/2 + (2+1) \\ &= 2(2+1)/2 + 2 \cdot (2+1)/2 \\ &= (2+2)(2+1)/2 = (3+1)(3)/2, \end{aligned}$$

  which is equal to $3(3+1)/2$.
- Line (*) is important here. That is because we use the fact that the statement is true when $n = 2$ there.

Informal arguments (3)

- Goal: show that $\sum_{i=1}^{n} i = n(n+1)/2$, for any integer $n \geq 1$.
- What we have just done? We show that the statement is true when $n = 3$ if it is true when $n = 2$.
- Let's try to make a more general argument.
- Assume that the statement is true for $n = k$. I.e.,

$$\sum_{i=1}^{k} i = k(k+1)/2.$$

- Can we show that, with this assumption, the statement is true for $n = k+1$? I.e., can we show that

$$\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2?$$

Informal arguments (4)

Let's try...

Assumption: $\sum_{i=1}^{k} i = k(k+1)/2$.

Goal: $\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2$.

$$
\begin{aligned}
\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + (k+1) \\
&= k(k+1)/2 + (k+1) \\
&= k(k+1)/2 + 2 \cdot (k+1)/2 \\
&= (k+2)(k+1)/2 \\
&= (k+1)((k+1)+1)/2,
\end{aligned}
$$

as required.

Informal arguments (5)

We have all the ingredients required to prove this statement:

> For integer $n \geq 1$, $\sum_{i=1}^{n} i = n \cdot (n+1)/2$.

Let $P(n) \equiv$ "$\sum_{i=1}^{n} i = n \cdot (n+1)/2$".
The statement we want to prove becomes:

> For any natural number n, $P(n)$.

We have shown:

1. $P(1)$ (by experimentation)
2. $P(k) \Rightarrow P(k+1)$ for any integer $k \geq 1$.

What do these two statements imply?

Informal arguments (6)

We have:

1. $P(1)$ (by experimentation)
2. $P(k) \Rightarrow P(k+1)$ for any integer $k \geq 1$.

What do these two statements imply?

$P(1)$ (1st statement itself)
$\Rightarrow P(2)$ (from 2nd statement, let $k = 1$)
$\Rightarrow P(3)$ (from 2nd statement, let $k = 2$)
$\Rightarrow P(4)$ (from 2nd statement, let $k = 3$)
$\Rightarrow P(5) \Rightarrow P(6) \Rightarrow P(7) \ldots$

Informally, these chain of reasoning will eventually reach any natural number n. Therefore, we can conclude that $P(n)$ for any natural number n.
We have just shown the statement with mathematical induction.

Mathematical Induction

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts:
Base case: $P(1)$
Inductive step: For any $k \geq 1$, $P(k) \Rightarrow P(k+1)$

The **Principle of Mathematical Induction** states that $P(n)$ is true for every natural number n.

The assumption $P(k)$ in the inductive step is usually referred to as **the Induction Hypothesis**.

Let's re-write the proof again

Theorem 1

For every natural number n, $\sum_{i=1}^{n} i = n(n+1)/2$

Proof: We prove by induction. The property that we want to prove $P(n)$ is "$\sum_{i=1}^{n} i = n(n+1)/2$."

Base case: We can plug in $n = 1$ to check that $P(1)$ is true: $1 = 1(1+1)/2$.

Inductive step: We assume that $P(k)$ is true for $k \geq 1$ and show that $P(k+1)$ is true.

Let's state the Induction Hypothesis $P(k)$: $\sum_{i=1}^{k} i = k(k+1)/2$.

Let's show $P(k+1)$. We write $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$. Using the Induction Hypothesis, we know that this is equal to

$$
\begin{aligned}
k(k+1)/2 + (k+1) &= k(k+1)/2 + 2 \cdot (k+1) \\
&= (k+2)(k+1)/2,
\end{aligned}
$$

which implies $P(k+1)$ as required.

From the Principle of Mathematical Induction, this implies that $P(n)$ is true for every natural number n.