01204211 Discrete Mathematics
Lecture 2b: Quantifiers

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Review (1)

- A *proposition* is a statement which is either **true** or **false**.
- We can use variables to stand for propositions, e.g., $P =$ "today is Tuesday".
- We can use connectives to combine variables to get propositional forms.
  - **Conjunction:** $P \wedge Q$ ("P and Q"),
  - **Disjunction:** $P \vee Q$ ("P or Q"), and
  - **Negation:** $\neg P$ ("not P")
  - **Implication:** $P \Rightarrow Q$ ("P implies Q", "if P, then Q", "P, only if Q")
  - **Equivalence:** $P \Leftrightarrow Q$ ("P if and only if Q")

Review (2): Testing primes

Consider the following code.

```
Algorithm CheckPrime2(n):  // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

How fast can it run? Note that $s = \sqrt{n}$; therefore, it takes time approximately proportional to $\sqrt{n}$ to run.

Ok, it should be faster. **But is it correct?**

The goals

- Let's recall what we are trying to do.

Original goal: To show that Algorithm `CheckPrime2` is correct.

Current (sub) goal: Consider a positive composite n and its positive divisor a, where $a > \sqrt{n}$. Let $b = n/a$. We want to show that $2 \leq b \leq \sqrt{n}$.

The (sub) goal

- **Current (sub) goal:** Consider a positive composite n and its positive divisor a, where $a > \sqrt{n}$. Let $b = n/a$. We want to show that $2 \leq b \leq \sqrt{n}$.
- We can be more specific about what values of n and b that we want to consider.

Revised statement

For all positive composite integer n, and for every divisor a of n such that $\sqrt{n} < a < n$,

$$2 \leq b \leq \sqrt{n},$$

where $b = n/a$.

- Note that this revised statement is now "quantified," that is, every variable in the statement has specific scope. Now the statement is either true or false.

Predicates

- In many cases, the statement we are interested in contains variables.
- For example, "x is even," "p is prime," or "s is a student."
- As we previously did with propositions, we can use variables to represent these statements. E.g.,
  - let $E(x) \equiv$ "x is even",
  - let $P(y) \equiv$ "y is prime, and
  - let $S(w) \equiv$ "w is a student.

  We call $E(x)$, $P(y)$, and $S(w)$ *predicates*. (You can think of predicates as statements that may be true of false depending on the values of its variables.)

Quantifiers (1)

- As we note before, these predicates are not propositions. But if we know the values of their variables, then they becomes propositions. For example, if we let $x = 5$, then $E(5)$ is a proposition which is false. Also, $P(7)$ is true.
- Since the truth values of predicates depend on the assignments of their variables, we can put *quantifiers* to specify the scopes of these variables and how to interprete the truth values of the predicates over these values.

Quantifiers (2): universal quantifiers

- Let $A = \{2, 4, 6, 8\}$.
- Note that $E(2), E(4), E(6)$, and $E(8)$ are true, i.e., $E(x)$ is true for every $x \in A$.
  In this case, we say that the following proposition is true:

$$(\forall x \in A)E(x).$$

- The quantifier $\forall$ is called a universal quantifier. (We usually pronounce "for all x", or "for every x.")

Quantifiers (3): existential quantifiers

- Again, let $A = \{2, 4, 6, 8\}$.
- Note that $P(2)$ is true. This means that $P(y)$ is true for some $y \in A$.
  In this case, we say that the following proposition is true:

$$(\exists y \in A)P(y).$$

- The quantifier $\exists$ is called an existential quantifier. (We usually pronounce "for some x", or "there exists x.")

> When the universe A is clear, we can leave it out and just write $\forall x E(x)$ or $\exists y P(y)$.

The main goal

- Let's try to be more specific about our main goal:

  Algorithm `CheckPrime2` is correct.

- Can we re-write this statement so that the input/output of the algorithm are explicit?
- Note that the set of its input n is an integer. Thus, we are interested in every $n \in \mathbb{Z}$, where $\mathbb{Z}$ denote the set of all integers.
- Let's rewrite the goal as:

  $$\forall n \in \mathbb{Z},\ C(n) \Leftrightarrow P(n),$$

  where $C(n) \equiv$ "`CheckPrime2(n)` returns True", and $P(n) \equiv$ "n is a prime."

Quantified propositions with more than one variables

Let our universe be integers ($\mathbb{Z}$). Which of the following statements is true?

- $\forall x \forall y (x = y)$
- $\forall x \exists y (x = y)$
- $\exists x \forall y (x = y)$
- $\exists x \exists y (x = y)$

When you have many quantifiers, we can interprete the statement by nesting the quantifiers. E.g,

$$\exists x \forall y P(x, y) \equiv \exists x (\forall y (P(x, y))).$$

$$\forall y \exists x P(x, y) \equiv \forall y (\exists x (P(x, y))).$$

Also note that usually, $\exists x \forall y P(x, y) \not\equiv \forall y \exists x P(x, y)$.

Quick check 4

We will consider the universe to be "everything". Consider the following statements. Define appropriate predicates and rewrite them using the defined predicates and quantifiers. (Note: the predicates may have more than one variables.)

- Every human must die.
- Some animal eats other animals.
- If a student works hard, that student will be successful.
- Everyone has someone that care about him or her.

Quick check 5

- Let's consider the current subgoal. (Note that in this version, variable b is replaced with n/a.)

Another revised statement

For all positive composite integer n, and for every divisor a of n such that $\sqrt{n} < a < n$,

$$2 \leq n/a \leq \sqrt{n}.$$

- Define all required predicates and describe a quantified proposition equivalent to the revised statement above.

Negations of quantified propositions (1)

Let consider a set of positive integers $\mathbb{Z}^+$ as our universe. Let predicate $P(x) \equiv$ "x is a prime number."
Consider this proposition

$$(\forall x \in \mathbb{Z}^+) P(x).$$

How can we show that this is false?
When showing that a universally quantified proposition is false, we need to show "one" counter example. In this case, since $P(4)$ is false, $\forall x P(x)$ is false.
This way of disproving a statement is equivalent to showing that

$$(\exists x)(\neg P(x)).$$

Negations of quantified propositions (2)

Let consider a set of positive integers $\mathbb{Z}^+$ as our universe. Let predicate $Q(x) \equiv$ "if $x > 2$, then $x^2 \leq 2x$."
Consider this proposition

$$(\exists x \in \mathbb{Z}^+) Q(x).$$

How can we show that this is false?
When showing that an existential quantified proposition is false, we need to show that $Q(x)$ is false for every possible values of x. In this case, since $x^2 = x \cdot x > 2 \cdot x$ for every $x > 2$, we have that $(\exists x) Q(x)$ is false.
This way of disproving a statement is equivalent to showing that

$$(\forall x)(\neg Q(x)).$$

Negations of quantified propositions (3)

Thus, the following equivalences:

- $\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$
- $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$

Quick check 6

Consider the following statements with the quantified propositions that you have written previously. Write down their negations in quantified propositional forms, and then translate them back to English sentences.

- Every human must die.
- Some animal eats other animals.
- If a student works hard, that student will be successful.
- Everyone has someone that care about him or her.