01204211 Discrete Mathematics
Lecture 1b: Implications and equivalences

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This lecture covers:

- More connectives: implications and equivalences

Review (1)

- A *proposition* is a statement which is either **true** or **false**.
- We can use variables to stand for propositions, e.g., $P =$ "today is Tuesday".
- We can use connectives to combine variables to get propositional forms.
  - **Conjunction:** $P \wedge Q$ ("P and Q"),
  - **Disjunction:** $P \vee Q$ ("P or Q"), and
  - **Negation:** $\neg P$ ("not P")

Review (2)

To represents values of propositional forms, we usually use truth tables.

And/Or/Not

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
T	T	T	T	F
T	F	F	T	
F	T	F	T	T
F	F	F	F	

Quick check 1

As we said before, the truth value of propositional forms may not depend on the values of its variables. As you can see in this exercise.

Use a truth table to find the values of (1) $P \wedge \neg P$ and (2) $P \vee \neg P$.

And/Or/Not

P	$\neg P$	$P \wedge \neg P$	$P \vee \neg P$
T	F	F	T
F	T	F	T

Note that $P \wedge \neg P$ is always false and $P \vee \neg P$ is always true.

A propositional form which is always true regardless of the truth values of its variables is called a *tautology*. On the other hand, a propositional form which is always false regardless of the truth values of its variables is called a *contradiction*.

Implications[1]

Given P and Q, an implication

$$P \Rightarrow Q$$

stands for "if P, then Q". This is a very important propositional form.
It states that "when P is true, Q must be true". Let's try to fill in its truth table:

Implications

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

[1] Materials in this lecture are mostly from Berkeley CS70's lecture notes.

What?

- Yes, when P is false, $P \Rightarrow Q$ is **always true** no matter what truth value of Q is.
- We say that in this case, the statement $P \Rightarrow Q$ is *vacuously true.*
- You might feel a bit uncomfortable about this, because in most natural languages, when we say that if P, then Q we sometimes mean something more than that in the logical expression "$P \Rightarrow Q$."

One explanation

- But let's look closely at what it means when we say that:

  > if P is true, Q must be true.

- Note that this statement does not say anything about the case when P is false, i.e., it only considers the case when P is true.
- Therefore, having that $P \Rightarrow Q$ is true is OK with the case that (1) Q is false when P is false, and (2) Q is true when P is false.
- This is an example when mathematical language is "stricter" than natural language.

Noticing if-then

We can write "if P, then Q" for $P \Rightarrow Q$, but there are other ways to say this. E.g., we can write (1) Q if P, (2) P only if Q, or (3) when P, then Q.

Quick check 2

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- If you do not have enough sleep, you will feel dizzy during class.
- If you eat a lot and you do not have enough exercise, you will get fat.
- You can get A from this course, only if you work fairly hard.

Only-if

Let P be "you get A from this course."
Let Q be "you work fairly hard."
Let R be "You can get A from this course, only if you work fairly hard."
Let's think about the truth values of R.

Only if you work fairly hard.

P	Q	R
T	T	
T	F	
F	T	
F	F	

Thus, R should be logically equivalent to $P \Rightarrow Q$. (We write $R \equiv P \Rightarrow Q$ in this case.)

If and only if: ($\Leftrightarrow$)

Given P and Q, we denote by

$$P \Leftrightarrow Q$$

the statement "P if and only if Q." It is logically equivalent to

$$(P \Leftarrow Q) \wedge (P \Rightarrow Q),$$

i.e., $P \Leftrightarrow Q \equiv (P \Leftarrow Q) \wedge (P \Rightarrow Q)$.
Let's fill in its truth table.

P	Q	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
T	T			
T	F			
F	T			
F	F			

An implication and its friends

When you have two propositions

- $P =$ "I own a cell phone", and
- $Q =$ "I bring a cell phone to class".

We have

- an implication $P \Rightarrow Q \equiv$
  "If I own a cell phone, I'll bring it to class",
- its **converse** $Q \Rightarrow P \equiv$
  "If I bring a cell phone to class, I own it", and
- its **contrapositive** $\neg Q \Rightarrow \neg P \equiv$
  "If I do not bring a cell phone to class, I do not own one".

Quick check 3

Let's consider the following truth table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
T	T			
T	F			
F	T			
F	F			

Do you notice any equivalence?

Right, $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$.