01204211 Discrete Mathematics Lecture 1b: Implications and equivalences

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June 28, 2022

This lecture covers:

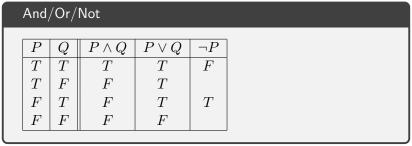
More connectives: implications and equivalences

Review (1)

- A *proposition* is a statement which is either **true** or **false**.
- We can use variables to stand for propositions, e.g., P = "today is Tuesday".
- We can use connectives to combine variables to get propositional forms.
 - Conjunction: $P \wedge Q$ ("*P* and *Q*"),
 - **Disjunction:** $P \lor Q$ ("*P* or *Q*"), and
 - Negation: $\neg P$ ("not P")



To represents values of propositional forms, we usually use truth tables.



Quick check 1

As we said before, the truth value of propositional forms may not depend on the values of its variables. As you can see in this exercise.

Use a truth table to find the values of (1) $P \land \neg P$ and (2) $P \lor \neg P$.

| | $And_{/}$ | /Or/N | ot | | |
|---|-----------|----------|-------------------|-----------------|--|
| [| P | $\neg P$ | $P \wedge \neg P$ | $P \vee \neg P$ | |
| | T | F | F | Т | |
| | F | T | F | T | |
| | | | | | |

Note that $P \land \neg P$ is always false and $P \lor \neg P$ is always true. A propositional form which is always true regardless of the truth values of its variables is called a *tautology*. On the other hand, a propositional form which is always false regardless of the truth values of its variables is called a *contradiction*.

Implications¹

Given P and Q, an implication

 $P \Rightarrow Q$

stands for "if P, then Q". This is a very important propositional form.

It states that "when ${\cal P}$ is true, ${\cal Q}$ must be true". Let's try to fill in its truth table:

| Impli | catic | ons | |
|---|---|---|--|
| $\begin{array}{ c c } P \\ \hline T \end{array}$ | Q T | $\begin{array}{c} P \Rightarrow Q \\ T \end{array}$ | |
| $\left \begin{array}{c} T \\ T \\ F \end{array}\right $ | $\begin{bmatrix} I \\ F \\ T \end{bmatrix}$ | F T | |
| $\begin{bmatrix} F \\ F \end{bmatrix}$ | $\begin{bmatrix} I \\ F \end{bmatrix}$ | $\begin{array}{c} I \\ T \end{array}$ | |

¹Materials in this lecture are mostly from Berkeley CS70's lecture notes.

What?

- Yes, when P is false, P ⇒ Q is always true no matter what truth value of Q is.
- We say that in this case, the statement $P \Rightarrow Q$ is vacuously true.
- You might feel a bit uncomfortable about this, because in most natural languages, when we say that if P, then Q we sometimes mean something more than that in the logical expression "P ⇒ Q."

One explanation

But let's look closely at what it means when we say that:

if P is true, Q must be true.

- Note that this statement does not say anything about the case when P is false, i.e., it only considers the case when P is true.
- ► Therefore, having that P ⇒ Q is true is OK with the case that (1) Q is false when P is false, and (2) Q is true when P is false.
- This is an example when mathematical language is "stricter" than natural language.

Noticing if-then

We can write "if P, then Q" for $P \Rightarrow Q$, but there are other ways to say this. E.g., we can write (1) Q if P, (2) P only if Q, or (3) when P, then Q.

Quick check 2

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- If you do not have enough sleep, you will feel dizzy during class.
- If you eat a lot and you do not have enough exercise, you will get fat.
- You can get A from this course, only if you work fairly hard.

Only-if

Let P be "you get A from this course."

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Let Q be "you work fairly hard."
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Let R be "You can get A from this course, only if you work fairly hard."

Let's think about the truth values of R.



Thus, R should be logically equivalent to $P \Rightarrow Q.$ (We write $R \equiv P \Rightarrow Q$ in this case.)

If and only if: (\Leftrightarrow)

Given P and Q, we denote by

 $P \Leftrightarrow Q$

the statement "P if and only if $Q."\,$ It is logically equivalent to

$$(P \Leftarrow Q) \land (P \Rightarrow Q),$$

i.e.,
$$P \Leftrightarrow Q \equiv (P \leftarrow Q) \land (P \Rightarrow Q)$$
.
Let's fill in its truth table.

| $\begin{array}{c c c c c c c c c c c c c c c c c c c $ |
|--|
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| |
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| |

An implication and its friends

When you have two propositions

- \blacktriangleright P = "I own a cell phone", and
- Q = "I bring a cell phone to class".

We have

- ▶ an implication $P \Rightarrow Q \equiv$ "If I own a cell phone, I'll bring it to class",
- its converse $Q \Rightarrow P \equiv$

"If I bring a cell phone to class, I own it", and

its contrapositive ¬Q ⇒ ¬P ≡
"If I do not bring a cell phone to class, I do not own one".

Quick check 3

Let's consider the following truth table:

| P | Q | $P \Rightarrow Q$ | $Q \Rightarrow P$ | $\neg Q \Rightarrow \neg P$ | |
|---|---|-------------------|-------------------|-----------------------------|--|
| T | T | | | | |
| T | F | | | | |
| F | T | | | | |
| F | F | | | | |

Do you notice any equivalence? Right, $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$.